Introduction to Algebraic Number Theory
ISI Bangalore
Semester I, 2015-2016
Mid-Semester Exam
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Remark: Minkowski's constant is $\frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s\sqrt{|\mathrm{disc}(K)|}$, where K is a number field of degree n with r real and s non-real embeddings.

(1) Determine with proof the class group of $\mathbb{Q}(\sqrt{-13})$. (Hint: For some idea of what it could be, see the next question.)

(2) Assume that every nonzero element of the class group of $\mathbb{Q}(\sqrt{-13})$ has order 2. Find all integer solutions to $y^2 = x^3 - 13$.

(3) Let R be a Dedekind domain with only finitely many prime ideals. Show that R is a PID, and give an example of such a ring.

(4) Recall that a valuation on a field K is a function $v : K^* \to \Gamma$ for some totally ordered abelian group, satisfying $v(xy) = v(x)+v(y)$, and $v(x+y) \geq \min\{v(x), v(y)\}$. Let $R = \{x \in K^* \mid v(x) \geq 0\} \cup \{0\}$. Show that
   (a) $v(1) = v(-1) = 0$, and more generally, $v(x) = 0$ if and only if x is an invertible element of R.
   (b) The set $M = \{x \in K^* \mid v(x) > 0\} \cup \{0\}$ is an ideal in R, and is in fact the unique maximal ideal (so R is a local ring).
   (c) Assume now that $\Gamma = \mathbb{Z}$. Show that M is principal, and that the ideals of R are precisely the ideals M^k, $k = 1, 2, \ldots$.

(5) Let g be odd, $g > 1$. Let x be odd, and satisfy $x^2 < 3^g/2$. Write d for $3^g - x^2$, and assume d is square free. Show that $\mathbb{Q}(\sqrt{-d})$ has an element in the class group of order g. (Hint: Write $3^g = (x + \sqrt{-d})(x - \sqrt{-d})$ and proceed exactly as you did in a similar homework problem.)